

# INVARIANT THEORY OF RELATIVELY FREE RIGHT-SYMMETRIC AND NOVIKOV ALGEBRAS

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*Dedicated to Askar Dzhumadil'daev on the occasion of his 60th birthday*

**ABSTRACT.** Algebras with the polynomial identity  $(x_1, x_2, x_3) = (x_1, x_3, x_2)$ , where  $(x_1, x_2, x_3) = x_1(x_2x_3) - (x_1x_2)x_3$  is the associator, are called right-symmetric. Novikov algebras are right-symmetric algebras satisfying additionally the polynomial identity  $x_1(x_2x_3) = x_2(x_1x_3)$ . We consider the free right-symmetric algebra  $F_d(\mathfrak{R})$  and the free Novikov algebra  $F_d(\mathfrak{N})$  freely generated by  $X_d = \{x_1, \dots, x_d\}$  over a field  $K$  of characteristic 0. The general linear group  $GL_d(K)$  with its canonical action on the  $d$ -dimensional vector space  $KX_d$  acts on  $F_d(\mathfrak{R})$  and  $F_d(\mathfrak{N})$  as a group of linear automorphisms. For a subgroup  $G$  of  $GL_d(K)$  we study the algebras of  $G$ -invariants  $F_d(\mathfrak{R})^G$  and  $F_d(\mathfrak{N})^G$ . For a large class of groups  $G$  we show that the algebras  $F_d(\mathfrak{R})^G$  and  $F_d(\mathfrak{N})^G$  are never finitely generated. The same result holds for any subvariety of the variety  $\mathfrak{R}$  of right-symmetric algebras which contains the subvariety  $\mathfrak{L}$  of left-nilpotent of class 3 algebras in  $\mathfrak{R}$ .

## INTRODUCTION

In this paper we fix a field  $K$  of characteristic 0 and consider nonassociative  $K$ -algebras. An algebra  $A$  is called *right-symmetric* if it satisfies the polynomial identity

$$(x_1, x_2, x_3) = (x_1, x_3, x_2), \quad (1)$$

where  $(x_1, x_2, x_3) = x_1(x_2x_3) - (x_1x_2)x_3$  is the associator, i.e.,

$$(a_1, a_2, a_3) = (a_1, a_3, a_2) \text{ for all } a_1, a_2, a_3 \in A.$$

A right-symmetric algebra is *Novikov* if it satisfies additionally the polynomial identity of left-commutativity

$$x_1(x_2x_3) = x_2(x_1x_3). \quad (2)$$

We denote by  $\mathfrak{R}$  and  $\mathfrak{N}$  the varieties of all right-symmetric algebras and all Novikov algebras, respectively. For details on the history of right-symmetric and Novikov algebras we refer to the introductions of the paper by Dzhumadil'daev and Löfwall [20] and the recent preprint by Bokut, Chen, and Zhang [4]. The origins of the right-symmetric algebras can be traced back till the paper by Cayley [6] in 1857. Translated in modern language, Cayley mentioned an identity which implies the

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2010 *Mathematics Subject Classification.* 17A36; 17A30; 17A50; 15A72.

*Key words and phrases.* Free right-symmetric algebras, free Novikov algebras, noncommutative invariant theory.

Partially supported by Grant I02/18 “Computational and Combinatorial Methods in Algebra and Applications” of the Bulgarian National Science Fund.

right-symmetric identity for the associators and holds for the right-symmetric Witt algebra in  $d$  variables

$$W_d^{\text{rsym}} = \left\{ \sum_{i=1}^d f_i \frac{\partial}{\partial x_i} \mid f_i \in K[X_d] \right\}$$

equipped with the multiplication

$$\left( f_i \frac{\partial}{\partial x_i} \right) * \left( f_j \frac{\partial}{\partial x_j} \right) = \left( f_j \frac{\partial f_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Cayley also considered the realization of  $W_d^{\text{rsym}}$  in terms of rooted trees. Later right-symmetric algebras were studied under different names: Vinberg, Koszul, Gerstenhaber, and pre-Lie algebras, see the references in [20]. The opposite algebras of Novikov algebras (satisfying the left-symmetric identity for the associators and right commutativity) appeared in the paper by Gel'fand and Dorfman [22]. There the authors gave an algebraic approach to the notion of Hamiltonian operator in finite-dimensional mechanics and the formal calculus of variations. Independently, later Novikov algebras were rediscovered by Balinskii and Novikov in the study of equations of hydrodynamics [2], see also the survey article by Novikov [31]. (Due to the contributions in [22, 2] and [31] Bokut, Chen, and Zhang [4] suggest to call these algebras Gel'fand-Dorfman-Novikov algebras. We shall continue to keep the name Novikov algebras.) An example of a Novikov algebra is the right-symmetric Witt algebra  $W_1^{\text{rsym}}$  in one variable. In a series of papers, see, e.g., [20, 18, 19] Dzhumadil'daev, with coauthors or alone, has studied free right-symmetric and free Novikov algebras, with applications to nonassociative algebras with polynomial identities.

In commutative invariant theory one usually considers the general linear group  $GL_d(K)$  with its canonical action on the  $d$ -dimensional vector space  $V_d$  with basis  $\{e_1, \dots, e_d\}$ . This induces an action on the polynomial algebra  $K[X_d] = K[x_1, \dots, x_d]$  in  $d$  variables

$$g(f(v)) = f(g^{-1}(v)), \quad g \in GL_d(K), v \in V_d,$$

where the linear functions  $x_i : V_d \rightarrow K$  are defined by

$$x_i(e_j) = \delta_{ij}, \quad i, j = 1, \dots, d,$$

and  $\delta_{ij}$  is the Kronecker symbol. For our noncommutative considerations it is more convenient to suppress one step and, replacing  $V$  with its dual space  $V^*$ , to assume that  $GL_d(K)$  acts canonically on the vector space  $KX_d$  with basis  $X_d = \{x_1, \dots, x_d\}$ . Then, identifying the polynomial algebra  $K[X_d]$  with the symmetric algebra of  $KX_d$ , we extend diagonally this action of  $GL_d(K)$  on  $K[X_d]$ :

$$g(f(X_d)) = g(f(x_1, \dots, x_d)) = f(g(x_1), \dots, g(x_d)), \quad (3)$$

$g \in GL_d(K)$ ,  $f(X_d) \in K[X_d]$ . In this way  $GL_d(K)$  acts as the group of linear automorphisms of  $K[X_d]$ . For a subgroup  $G$  of  $GL_d(K)$  the algebra of  $G$ -invariants is

$$K[X_d]^G = \{f \in K[X_d] \mid g(f) = f \text{ for all } g \in G\}.$$

This is a  $\mathbb{Z}$ -graded vector space and its *Hilbert* (or *Poincaré*) *series* is the formal power series

$$H(K[X_d]^G, z) = \sum_{n \geq 0} \dim(K[X_d]^G)_n z^n,$$

where  $(K[X_d]^G)_n$  is the homogeneous component of degree  $n$  in  $K[X_d]^G$ . The following are among the main problems related with the description of the algebra  $K[X_d]^G$  for different groups or classes of groups  $G$ . For details concerning also computational and algorithmic problems see [9] or [35].

- *Is the algebra  $K[X_d]^G$  finitely generated?* This problem was the main motivation for the Hilbert 14th problem in his famous lecture “*Mathematische Probleme*” given at the International Congress of Mathematicians held in 1900 in Paris [25]. It is known that  $K[X_d]^G$  is finitely generated for finite groups (the theorem of Emmy Noether [30]), for reductive groups (the Hilbert-Nagata theorem, see e.g., [11]), and for groups close to reductive (see e.g., Grosshans [23] and Hadžiev [24]). The first example of an algebra of invariants  $K[X_d]^G$  which is not finitely generated is due to Nagata [29].
- *If  $K[X_d]^G$  is finitely generated, describe it in terms of generators and defining relations.* In different degree of generality this problem is solved for classes of groups. For example, the theorem of Emmy Noether [30] gives that for finite groups the algebra  $K[X_d]^G$  is generated by invariants of degree  $\leq |G|$ . Also for finite groups, the Chevalley-Shephard-Todd theorem [7, 33] states that the algebra  $K[X_d]^G$  is isomorphic to the polynomial algebra in  $d$  variables (i.e., it is generated by a set of  $d$  algebraically independent invariants) if and only if  $G$  is generated by pseudo-reflections.
- *Calculate the Hilbert series  $H(K[X_d]^G, z)$ .* For finite groups the answer is given by the Molien formula [28]

$$H(K[X_d]^G, z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gz)}.$$

The analogue for reductive and close to them groups is the Molien-Weyl integral formula [37], see also [38].

In noncommutative invariant theory one replaces the polynomial algebras  $K[X_d]$  with other noncommutative or nonassociative algebras still keeping some of the typical features of polynomial algebras. One such feature is the universal property that for an arbitrary commutative algebra  $A$  every mapping  $X_d \rightarrow A$  is extended to a homomorphism  $K[X_d] \rightarrow A$ . In the noncommutative set-up the class of commutative algebras is replaced by an arbitrary variety of algebras  $\mathfrak{V}$  and instead on  $K[X_d]^G$  one studies the algebra of  $G$ -invariants  $F_d(\mathfrak{V})^G$  of the  $d$ -generated relatively free algebra  $F_d(\mathfrak{V})$  in  $\mathfrak{V}$ ,  $d \geq 2$ . For a background see the surveys [21, 13]. Comparing with commutative invariant theory, when  $K[X_d]^G$  is finitely generated for all “nice” groups, the main difference in the noncommutative case is that  $F_d(\mathfrak{V})^G$  is finitely generated quite rarely. For a survey on invariants of finite groups  $G$  acting on relatively free associative algebras see [21, 13] and [26]. For finite groups  $G \neq \langle 1 \rangle$  and varieties of Lie algebras  $F_d(\mathfrak{V})^G$  is finitely generated if and only if  $\mathfrak{V}$  is nilpotent, see [5, 12].

Concerning the Hilbert series of  $F_d(\mathfrak{V})^G$ , for  $G$  finite there is an analogue of the Molien formula, see Formanek [21]. Let

$$H(F_d(\mathfrak{V}), z_1, \dots, z_d) = \sum_{n_i \geq 0} \dim F_d(\mathfrak{V})_{(n_1, \dots, n_d)} z_1^{n_1} \cdots z_d^{n_d}$$

be the Hilbert series of  $F_d(\mathfrak{V})$  as a multigraded vector space. It is equal to the generating function of the dimensions of the vector spaces  $F_d(\mathfrak{V})_{(n_1, \dots, n_d)}$  of the

elements in  $F_d(\mathfrak{V})$  which are homogeneous of degree  $n_i$  in  $x_i$ . If  $\xi_1(g), \dots, \xi_d(g)$  are the eigenvalues of  $g \in G$ , then the Hilbert series of the algebra of invariants  $F_d(\mathfrak{V})^G$  is

$$H(F_d(\mathfrak{V})^G; z) = \frac{1}{|G|} \sum_{g \in G} H(F_d(\mathfrak{V}); \xi_1(g)z, \dots, \xi_d(g)z).$$

There is also an analogue of the Molien-Weyl formula for the Hilbert series of  $F_d(\mathfrak{V})^G$  which combines ideas of De Concini, Eisenbud, and Procesi [8] and Almkvist, Dicks, and Formanek [1]. Evaluating the corresponding multiple integral one uses the Hilbert series of  $F_d(\mathfrak{V})$  instead of the Hilbert series of  $K[X_d]$

$$H(K[X_d], z_1, \dots, z_d) = \prod_{i=1}^d \frac{1}{1 - z_i}.$$

We refer to [3] for other methods for computing the Hilbert series of  $F_d(\mathfrak{V})^G$  when  $G$  is isomorphic to the special linear group  $SL_m(K)$  or to the group  $UT_m(K)$  of the  $m \times m$  unitriangular matrices.

In this paper we study invariant theory of relatively free right-symmetric and Novikov algebras. Let  $\mathfrak{L}$  be the variety of right-symmetric algebras which are left-nilpotent of class 3, i.e.,  $\mathfrak{L}$  is the subvariety of  $\mathfrak{R}$  satisfying the polynomial identity

$$x_1(x_2x_3) = 0. \quad (4)$$

For a large class of subgroups  $G$  of  $GL_d(K)$ ,  $G \neq \langle 1 \rangle$ ,  $d > 1$ , we show that  $F_d(\mathfrak{V})^G$  is not finitely generated for any variety  $\mathfrak{V}$  containing  $\mathfrak{L}$ . More precisely, let  $A_d = K[X_d]_+$  be the algebra of polynomials without constant term and let  $(A_d)_1^G = (KX_d)^G$  be the vector space of linear polynomials fixed by  $G$ . Clearly,  $(A_d)_1^G$  is a  $K[(A_d)_1^G]$ -module. If  $(A_d)_1^G$  is not finitely generated as a  $K[(A_d)_1^G]$ -module, then  $F_d(\mathfrak{V})^G$  is not finitely generated for any  $\mathfrak{V}$  containing  $\mathfrak{L}$ . The class of such groups  $G$  contains all finite groups. It contains also the classical and close to them groups under some natural restrictions on the embedding into  $GL_d(K)$ . In particular, if  $(A_d)_1^G = 0$  and  $(A_d)_2^G \neq 0$ , then  $F_d(\mathfrak{V})^G$  is not finitely generated. Results in the same spirit hold if we replace the polynomial algebra  $K[X_d]$  with the free metabelian Lie algebra  $F_d(\mathfrak{A}^2) = L_d/L_d''$ , where  $L_d$  is the free Lie algebra freely generated by  $X_d$  and  $\mathfrak{A}^2$  is the variety of all metabelian (solvable of class 2) Lie algebras. If  $(KX_d)^G = (A_d)_1^G = 0$  and  $\dim F_d(\mathfrak{A}^2)^G = \infty$ , then again  $F_d(\mathfrak{V})^G$  is not finitely generated.

## 1. PRELIMINARIES

We fix a field  $K$  of characteristic 0. All vector spaces and algebras will be over  $K$ . Let

$$F(X) = K\{X\} = K\{x_1, x_2, \dots\}$$

be the (absolutely) free nonassociative algebra freely generated by the countable set  $X = \{x_1, x_2, \dots\}$ . Recall that the polynomial  $f(x_1, \dots, x_m) \in K\{X\}$  is a polynomial identity for the algebra  $A$  if  $f(a_1, \dots, a_m) = 0$  for all  $a_1, \dots, a_m \in A$ . The class of all algebras satisfying a given set  $U \subset K\{X\}$  of polynomial identities is called the variety of associative algebras defined by the system  $U$ . If  $\mathfrak{V}$  is a variety, then  $T(\mathfrak{V})$  is the ideal of  $K\{X\}$  consisting of all polynomial identities of  $\mathfrak{V}$ . Let  $X_d = \{x_1, \dots, x_d\} \subset X$ . Then the algebra

$$F_d(\mathfrak{V}) = K\{x_1, \dots, x_d\} / (K\{x_1, \dots, x_d\} \cap T(\mathfrak{V})) = K\{X_d\} / (K\{X_d\} \cap T(\mathfrak{V}))$$

is the relatively free algebra of rank  $d$  in  $\mathfrak{V}$ . We shall denote the generators of  $F_d(\mathfrak{V})$  with the same symbols  $X_d$ . The ideals  $K\{X_d\} \cap T(\mathfrak{V})$  of  $K\{X_d\}$  are preserved by all endomorphisms  $\varphi$  of  $K\{X_d\}$ , i.e.,  $\varphi(K\{X_d\} \cap T(\mathfrak{V})) \subseteq K\{X_d\} \cap T(\mathfrak{V})$ . In particular,  $GL_d(K)(K\{X_d\} \cap T(\mathfrak{V})) = K\{X_d\} \cap T(\mathfrak{V})$ . Here the general linear group  $GL_d(K)$  acts canonically on the vector space  $KX_d$  with basis  $X_d$  and this action is extended diagonally on the whole  $F_d(\mathfrak{V})$  as in (3). Hence  $F_d(\mathfrak{V})$  has a natural structure of a  $GL_d(K)$ -module. For a background on representation theory of  $GL_d(K)$  see, e.g., [27, 38]. Since  $\text{char}(K) = 0$ , the algebra  $F_d(\mathfrak{V})$  is a direct sum of irreducible  $GL_d(K)$ -modules and

$$F_d(\mathfrak{V}) = \sum m_\lambda(\mathfrak{V}) W_d(\lambda),$$

where  $W_d(\lambda)$  is the irreducible polynomial  $GL_d(K)$ -module corresponding to the partition  $\lambda = (\lambda_1, \dots, \lambda_d)$ ,  $\lambda_1 \geq \dots \geq \lambda_d \geq 0$ , and  $m_\lambda(\mathfrak{V})$  is the multiplicity of  $W_d(\lambda)$  in the decomposition of  $F_d(\mathfrak{V})$ . Then the Hilbert series of  $F_d(\mathfrak{V})$  is

$$H(F_d(\mathfrak{V}), z_1, \dots, z_d) = \sum m_\lambda(\mathfrak{V}) S_\lambda(z_1, \dots, z_d),$$

where  $S_\lambda(z_1, \dots, z_d)$  is the Schur function corresponding to  $\lambda$ . Since the Schur functions form a basis of the vector space  $K[X_d]^{S_n}$  of symmetric polynomials in  $d$  variables, the Hilbert series  $H(F_d(\mathfrak{V}), z_1, \dots, z_d)$  determines the  $GL_d(K)$ -module structure of  $F_d(\mathfrak{V})$ .

In the sequel we shall need some well known information for two relatively free algebras: the polynomial algebra  $K[X_d]$  and the free metabelian Lie algebra  $F_d(\mathfrak{A}^2) = L_d/L_d''$ .

**Lemma 1.1.** (i) *The  $GL_d(K)$ -module structure of the polynomial algebra  $K[X_d]$  is*

$$K[X_d] = \sum_{n \geq 0} W_d(n).$$

(ii) *The free metabelian Lie algebra  $F_d(\mathfrak{A}^2)$  has a basis*

$$\{x_i, [\dots [x_{i_1}, x_{i_2}], \dots], x_{i_n}] \mid i, i_j = 1, \dots, d, i_1 > i_2 \leq \dots \leq i_n\}.$$

*The  $GL_d(K)$ -module structure of  $F_d(\mathfrak{A}^2)$  is*

$$F_d(\mathfrak{A}^2) = W_d(1) + \sum_{n \geq 2} W_d(n-1, 1).$$

Part (i) of the lemma is well known. Part (ii) is also well known, see e.g., [34, §52, pp. 274-276 of the English translation] for the basis of  $F_d(\mathfrak{A}^2)$  and [17, the proof of Lemma 2.5] for its  $GL_d(K)$ -module structure.

The product of two Schur functions  $S_\lambda(z_1, \dots, z_d) S_\mu(z_1, \dots, z_d)$  can be expressed as a sum of Schur functions using the Littlewood-Richardson rule. A very special case of this rule is the Branching theorem, when  $\mu = (1)$ . It states that

$$S_\lambda(z_1, \dots, z_d) S_{(1)}(z_1, \dots, z_d) = \sum S_\nu(z_1, \dots, z_d), \quad (5)$$

where the sum runs on all partitions  $\nu = (\nu_1, \dots, \nu_d)$  obtained by adding 1 to one of the components  $\lambda_i$  of  $\lambda = (\lambda_1, \dots, \lambda_d)$ . In other words, the Young diagram of  $\nu$  is obtained by adding a box to the diagram of  $\lambda$ . Since the product of two Schur functions corresponds to the tensor product of the corresponding irreducible  $GL_d(K)$ -modules, we obtain equivalently

$$W_d(\lambda) \otimes_K W_d(1) = \sum W_d(\nu), \quad (6)$$

with the same summation on  $\nu$  as in (5).

If  $G$  is a subgroup of  $GL_d(K)$ , then the  $GL_d(K)$ -action on the irreducible  $GL_d(K)$ -module  $W_d(\lambda)$  induces a  $G$ -action on  $W_d(\lambda)$ . Let  $W_d(\lambda)^G$  be the vector space of the elements of  $W_d(\lambda)$  fixed by  $G$ , i.e., of the  $G$ -invariants of  $W_d(\lambda)$ . If  $W$  is a graded  $GL_d(K)$ -module with polynomial homogeneous components,

$$W = \bigoplus_{k \geq 0} W_k, \quad W_k = \sum_{\lambda} m_{\lambda}(k) W_d(\lambda), \quad (7)$$

then its Hilbert series is

$$\begin{aligned} H(W, z_1, \dots, z_d, z) &= \sum_{k \geq 0} \left( \sum_{n_i \geq 0} \dim(W_k)_{(n_1, \dots, n_d)} z_1^{n_1} \dots z_d^{n_d} \right) z^k \\ &= \sum_{k \geq 0} \sum_{\lambda} m_{\lambda}(k) S_{\lambda}(z_1, \dots, z_d) z^k. \end{aligned} \quad (8)$$

**Lemma 1.2.** *Let  $W$  be a graded  $GL_d(K)$ -module with polynomial homogeneous components, as in (7), and let  $G$  be a subgroup of  $GL_d(K)$ . Then the Hilbert series of the  $G$ -invariants of  $W$*

$$H(W^G, z) = \sum_{k \geq 0} \dim W_k^G z^k$$

*is determined from the Hilbert series (8) of  $W$ .*

*Proof.* We follow the main ideas of the recent preprint [10] which contains more applications in the spirit of the lemma. Since the dimension of  $W_d(\lambda)^G$  depends on  $W_d(\lambda)$  only, and the Schur functions  $S_{\lambda}(z_1, \dots, z_d)$  are in 1-1 correspondence with the modules  $W_d(\lambda)$ , we conclude that  $\dim W_d(\lambda)^G$  is a function of  $S_{\lambda}(z_1, \dots, z_d)$ . This immediately completes the proof because

$$H(W^G, z) = \sum_{k \geq 0} \left( \sum_{\lambda} m_{\lambda}(k) \dim W_d(\lambda)^G \right) z^k.$$

□

The proof of the next statement can be found in [16, Proposition 4.2] in the case of homomorphic images of the free associative algebra  $K\langle X_d \rangle$ . The proof in the case below is exactly the same.

**Proposition 1.3.** *Let  $I$  be an ideal of the relatively free algebra  $F_d(\mathfrak{W})$  of the variety  $\mathfrak{W}$  and let  $I$  be preserved under the  $GL_d(K)$ -action on  $F_d(\mathfrak{W})$ . If  $G$  is a subgroup of  $GL_d(K)$ , then every  $G$ -invariant of the factor algebra  $F_d(\mathfrak{W})/I$  can be lifted to a  $G$ -invariant of  $F_d(\mathfrak{W})$ , i.e., under the canonical homomorphism*

$$\pi : F_d(\mathfrak{W}) \rightarrow F_d(\mathfrak{W})/I$$

*$F_d(\mathfrak{W})^G$  maps onto  $(F_d(\mathfrak{W})/I)^G$ . In particular, if  $\mathfrak{V}$  is a subvariety of the variety  $\mathfrak{W}$  and  $\pi : F_d(\mathfrak{W}) \rightarrow F_d(\mathfrak{V})$ , then*

$$\pi(F_d(\mathfrak{W})^G) = F_d(\mathfrak{V})^G.$$

For more details on varieties of algebras (in the associative case) and the applications of representation theory of  $GL_d(K)$  to PI-algebras we refer to the book [14].

## 2. THE MAIN RESULT

Let  $\mathfrak{L}$  be the subvariety of the variety of right-symmetric algebras  $\mathfrak{R}$  defined by the identity (4) of left-nilpotency of class 3. Since the identity (2) of left-commutativity is a consequence of (4),  $\mathfrak{L}$  is also a subvariety of the variety  $\mathfrak{N}$  of Novikov algebras. Working in  $\mathfrak{L}$ , the only nonzero products are left-normed. We shall omit the parentheses and shall write  $a_1 a_2 \cdots a_n$  instead of  $(\cdots (a_1 a_2) \cdots) a_n$  and  $a_1 a_2^k$  instead of  $a_1 \underbrace{a_2 \cdots a_2}_{k \text{ times}}$ .

**Lemma 2.1.** (i) *The relatively free algebra  $F_d(\mathfrak{L})$  has a basis*

$$\{x_{i_1} x_{i_2} \cdots x_{i_n} \mid i_1 = 1, \dots, d, \quad 1 \leq i_2 \leq \cdots \leq i_n \leq d\}. \quad (9)$$

(ii) *The  $GL_d(K)$ -module structure of  $F_d(\mathfrak{L})$  is*

$$F_d(\mathfrak{L}) = W_d(1) + \sum_{n \geq 2} (W_d(n) + W_d(n-1, 1)). \quad (10)$$

*Proof.* (i) Modulo the identity (4) the right-symmetric identity (1) reduces to

$$x_1 x_2 x_3 = x_1 x_3 x_2. \quad (11)$$

Hence  $\mathfrak{L}$  satisfies the identity

$$x_1 x_2 \cdots x_n = x_1 x_{\sigma(2)} \cdots x_{\sigma(n)}, \quad \sigma \in S_n, \sigma(1) = 1,$$

and the algebra  $F_d(\mathfrak{L})$  is spanned as a vector space on the elements (9). In order to show that (9) is a basis of  $F_d(\mathfrak{L})$  it is sufficient to construct an algebra  $A$  in  $\mathfrak{L}$  which is generated by  $a_1, \dots, a_d$  and has a basis

$$\{a_i a_1^{n_1} \cdots a_d^{n_d} \mid i_1 = 1, \dots, d, \quad n_j \geq 0\}. \quad (12)$$

Since  $A$  is a homomorphic image of  $F_d(\mathfrak{L})$ , this would imply that (9) is a basis of  $F_d(\mathfrak{L})$ . Consider the vector space  $A$  with basis (12) and define a multiplication there by the rule

$$(a_i a_1^{n_1} \cdots a_d^{n_d}) * a_j = a_i a_1^{n_1} \cdots a_j^{n_j+1} \cdots a_d^{n_d},$$

$$(a_i a_1^{n_1} \cdots a_d^{n_d}) * (a_i a_1^{m_1} \cdots a_d^{m_d}) = 0, \text{ if } m_j > 0 \text{ for some } j.$$

Obviously  $A$  satisfies the identities (4) and (11), and hence belongs to  $\mathfrak{L}$ .

(ii) For  $n \geq 2$  we divide the basis elements from (9) in two groups. The first group contains the monomials  $x_{i_1} x_{i_2} \cdots x_{i_n}$  with  $i_1 \leq i_2$  and the second group the monomials with  $i_1 > i_2$ . Obviously, the monomials in the first group are in 1-1 correspondence with the monomials of degree  $\geq 2$  in  $K[X_d]$ . By Lemma 1.1 (ii), the same holds for the monomials from the second group and the elements of degree  $\geq 2$  in  $F_d(\mathfrak{A}^2)$ . Hence the Hilbert series of  $F_d(\mathfrak{L})$  is a sum of the Hilbert series of the algebra of polynomials without constant term and the commutator ideal of the Lie algebra  $F_d(\mathfrak{A}^2)$ . Now the proof follows from Lemma 1.1.  $\square$

The construction in the proof of Lemma 2.1 (i) suggests that the algebra  $F_d(\mathfrak{L})$  has the structure of a right  $K[X_d]$ -module with action defined by

$$(x_p x_1^{n_1} \cdots x_d^{n_d}) \circ (x_1^{m_1} \cdots x_d^{m_d}) = x_p x_1^{n_1+m_1} \cdots x_d^{n_d+m_d}, \quad p = 1, \dots, d, n_j, m_j \geq 0.$$

Clearly, the ideal  $F_d^2(\mathfrak{L})$  of the elements in  $F_d(\mathfrak{L})$  without linear term is a  $K[X_d]$ -submodule. We shall denote by  $(A_d)_1$  the vector space  $KX_d$  and shall identify  $K[X_d]$  and  $K[(A_d)_1]$ .

The following theorem and its consequences together with the examples in the next section are the main results of the paper.

**Theorem 2.2.** *Let  $\mathfrak{V}$  be a subvariety of the variety  $\mathfrak{R}$  of all right-symmetric algebras and let  $\mathfrak{V}$  contain the variety  $\mathfrak{L}$  of left-nilpotent of class 3 algebras in  $\mathfrak{R}$ . If  $G \neq \langle 1 \rangle$  is a subgroup of  $GL_d(K)$  such that the ideal  $F_d^2(\mathfrak{L})^G$  of the algebra of invariants  $F_d(\mathfrak{L})^G$  is not finitely generated as a  $K[(A_d)_1^G]$ -module, then the algebra of  $G$ -invariants  $F_d(\mathfrak{V})^G$  is not finitely generated.*

*Proof.* By Proposition 1.3 the canonical homomorphism  $F_d(\mathfrak{V}) \rightarrow F_d(\mathfrak{L})$  maps  $F_d(\mathfrak{V})^G$  onto  $F_d(\mathfrak{L})^G$  and if  $F_d(\mathfrak{V})^G$  is finitely generated, the same is  $F_d(\mathfrak{L})^G$ . Hence it is sufficient to show that  $F_d(\mathfrak{L})^G$  is not finitely generated. Therefore we may work in  $F_d(\mathfrak{L})$  and assume that  $F_d(\mathfrak{L})^G$  is finitely generated. As a vector space  $F_d(\mathfrak{L})^G$  is a direct sum of the invariants of first degree  $(KX_d)^G = (A_d)_1^G$  and the invariants  $F_d^2(\mathfrak{L})^G$  without linear term. We may assume that  $F_d(\mathfrak{L})^G$  is generated by  $U = \{u_1, \dots, u_k\} \subset (A_d)_1^G$  and  $W = \{w_1, \dots, w_l\} \subset F_d^2(\mathfrak{L})^G$ . Since  $F_d(\mathfrak{L})^G F_d^2(\mathfrak{L})^G = 0$ , the only nonzero products of the generators of  $F_d(\mathfrak{L})^G$  are  $u_p u_{i_1} \cdots u_{u_m}$ , and  $w_q u_{i_1} \cdots u_{u_m}$ ,  $m \geq 0$ . Hence  $KU = (A_d)_1^G$ ,

$$F_d^2(\mathfrak{L})^G = \sum_{i=1}^k u_{p_i} u_{p_2} \circ K[(A_d)_1^G] + \sum_{j=1}^l w_q \circ K[(A_d)_1^G]$$

and  $F_d^2(\mathfrak{L})^G$  is a finitely generated  $K[(A_d)_1^G]$ -module which is a contradiction.  $\square$

**Corollary 2.3.** *Let  $A_d = K[X_d]_+$  be the algebra of polynomials without constant term and let  $G$  be a subgroup of  $GL_d(K)$ . If  $(A_d^2)^G$  is not finitely generated as a  $K[(A_d)_1^G]$ -module, then  $F_d(\mathfrak{V})^G$  is not finitely generated for any variety  $\mathfrak{V}$  containing  $\mathfrak{L}$ .*

*Proof.* By the Branching theorem (6)

$$W_d(n-1, 1) \otimes_K W_d(1) = W_d(n, 1) \oplus W_d(n-1, 2) \oplus W_d(n-1, 1, 1). \quad (13)$$

Consider the  $GL_d(K)$ -module decomposition of  $F_d(\mathfrak{L})$  given in Lemma 2.1 (ii). Since  $F_d(\mathfrak{L}) F_d^2(\mathfrak{L}) = 0$ , the only nonzero products  $W_d(\lambda) W_d(\mu)$  with  $\lambda$  or  $\mu$  equal to  $(n-1, 1)$ ,  $n \geq 2$ , come from

$$W_d(n-1, 1) W_d(1) = W_d(n-1, 1) F_d(\mathfrak{L}) = W_d(n-1, 1) (KX_d) = W_d(n-1, 1) (A_d)_1.$$

This is a homomorphic image in  $F_d(\mathfrak{L})$  of  $W_d(n-1, 1) \otimes_K W_d(1)$ . By (13) we derive that  $W_d(n-1, 1) F_d(\mathfrak{L}) \subset W_d(n, 1)$ . This implies that

$$I = \sum_{n \geq 2} W_d(n-1, 1) \subset F_d(\mathfrak{L})$$

is an ideal of  $F_d(\mathfrak{L})$  and the  $GL_d(K)$ -module structure of the factor algebra is

$$F_d(\mathfrak{L})/I = \sum_{n \geq 1} W_d(n) \cong A_d.$$

Hence the algebras  $F_d(\mathfrak{L})/I$  and  $A_d$  have the same Hilbert series and by Lemma 1.2 the same holds for their algebras of invariants. Since  $(A_d^2)^G$  is not finitely generated as a  $K[(A_d)_1^G]$ -module, the same is true for the  $K[(A_d)_1^G]$ -module  $F_d^2(\mathfrak{L})/I$ . By Proposition 1.3, the  $K[(A_d)_1^G]$ -module  $F_d^2(\mathfrak{L})$  is not finitely generated and the application of Theorem 2.2 completes the proof.  $\square$



**Corollary 2.4.** *Let  $F_d(\mathfrak{A}^2)$  be the free metabelian Lie algebra and let  $G$  be a subgroup of  $GL_d(K)$ . If  $(KX_d)^G = (A_d)_1^G = 0$  and  $\dim F_d(\mathfrak{A}^2)^G = \infty$ , then  $F_d(\mathfrak{V})^G$  is not finitely generated for any variety  $\mathfrak{V}$  containing  $\mathfrak{L}$ .*

*Proof.* Since  $(KX_d)^G = (A_d)_1^G = 0$  we obtain that  $F_d(\mathfrak{L})^G = F_d^2(\mathfrak{L})^G$ . Hence the algebra  $F_d(\mathfrak{L})^G$  is with trivial multiplication and the finite generation is equivalent to the finite dimensionality. As a  $GL_d(K)$ -module  $F_d(\mathfrak{A}^2)$  is a homomorphic image of  $F_d(\mathfrak{L})$ . Hence the vector space  $F_d(\mathfrak{A}^2)^G$  is a homomorphic image of  $F_d(\mathfrak{L})^G$ . This implies that  $\dim F_d^2(\mathfrak{L})^G = \infty$ , i.e., both the algebras  $F_d(\mathfrak{L})^G$  and  $F_d(\mathfrak{V})^G$  are not finitely generated.  $\square$

**Remark 2.5.** In Corollary 2.4 we cannot remove directly the restriction  $(KX_d)^G = 0$ , as in Corollary 2.3, because the  $GL_d(K)$ -submodule  $I = \sum_{n \geq 2} W_d(n)$  of  $F_d(\mathfrak{L})$  is not an ideal. For example, one can show that  $W_d(2)(KX_d) = W_d(3) \oplus W_d(2, 1)$ . Hence we cannot use the property that the Lie algebra  $F_d(\mathfrak{A}^2)^G$  is not finitely generated to show that the algebra  $F_d(\mathfrak{L})$  is also not finitely generated. On the other hand, we do not know examples of groups  $G$  when  $(KX_d)^G = 0$ ,  $K[X_d]^G = K$ , and  $\dim F_d(\mathfrak{A}^2)^G = \infty$ . Such an example would show that we may apply Corollary 2.4 when we cannot apply Corollary 2.3.

### 3. EXAMPLES

All examples in this section use the following statement which is a consequence of Corollary 2.3.

**Proposition 3.1.** *If for a subgroup  $G$  of  $GL_d(K)$*

$$\text{transcend.deg}(K[X_d]^G) > \dim(KX_d)^G,$$

*then the algebra  $F_d(\mathfrak{V})^G$  is not finitely generated for any variety  $\mathfrak{V}$  containing  $\mathfrak{L}$ .*

*Proof.* Let  $t = \text{transcend.deg}(K[X_d]^G)$ . Since  $K[X_d]^G$  is graded, we may choose  $t$  algebraically independent homogeneous elements in  $A_d^G = (K[X_d]_+)^G$ . If  $m = \dim(KX_d)^G$ , changing linearly the variables  $X_d$  we assume that  $(KX_d)^G$  has a basis  $X_m = \{x_1, \dots, x_m\}$  and  $K[(A_d)_1^G] = K[X_m]$ . Since  $t > m$ , we obtain that  $((A_d)^2)^G$  contains an element  $f(X_d)$ , such that the system  $X_m \cup \{f(X_d)\}$  is algebraically independent. Hence the  $K[X_m]$ -module generated by the powers  $f^k(X_d)$ ,  $k = 1, 2, \dots$ , is not finitely generated. Now the proof follows from Corollary 2.3.  $\square$

#### 3.1. Finite groups.

**Theorem 3.2.** *Let  $G$  be a finite subgroup of  $GL_d(K)$  and  $G \neq \langle 1 \rangle$ . Then the algebra  $F_d(\mathfrak{V})^G$  is not finitely generated for any variety  $\mathfrak{V}$  containing  $\mathfrak{L}$ .*

*Proof.* It is well known that for a finite group  $G$

$$\text{transcend.deg}(K[X_d]^G) = \text{transcend.deg}(K[X_d]) = d. \quad (14)$$

For self-containedness of the exposition, every element  $f(X_d) \in K[X_d]$  satisfies the equation

$$u_f(z) = \prod_{g \in G} (z - g(f(X_d))) = z^{|G|} - c_1 z^{|G|-1} + c_2 z^{|G|-2} - \dots \pm c_{|G|}$$

where the coefficients  $c_k$  are equal to the elementary symmetric polynomials in  $\{g(f(X_d)) \mid g \in G\}$ . Hence  $c_k \in K[X_d]^G$  and as a  $K[X_d]^G$ -module  $K[X_d]$  is generated by

$$x_1^{a_1} \cdots x_d^{a_d}, \quad 0 \leq a_i < |G|.$$

The finite generation of the  $K[X_d]^G$ -module  $K[X_d]$  implies (14) and the theorem follows from Proposition 3.1.  $\square$

**3.2. Reductive groups.** If  $G \subset GL_d(K)$  is a reductive group then there exists a  $G$ -submodule  $W$  of  $KX_d$  such that  $KX_d = (KX_d)^G \oplus W$ .

**Proposition 3.3.** *In the above notation, if  $K[W]^G \neq K$ , then  $F_d(\mathfrak{V})^G$  is not finitely generated for all  $\mathfrak{V}$  containing  $\mathfrak{L}$ .*

*Proof.* Since the elements of  $K[W]$  cannot be expressed as polynomials in  $(KX_d)^G$ , the condition  $K[W]^G \neq K$  implies that

$$\text{transcend.deg}(K[X_d]^G) = \dim(KX_d)^G + \text{transcend.deg}(K[W]^G) > \dim(KX_d)^G$$

and this completes the proof in virtue of Proposition 3.1.  $\square$

**Example 3.4.** For each  $k \geq 1$  there is a unique irreducible rational  $k$ -dimensional  $SL_2(K)$ -module  $W_k$ . Let the subgroup  $G$  of  $GL_d(K)$  be isomorphic to  $SL_2(K)$  and

$$KX_d \cong W_{k_1} \oplus \cdots \oplus W_{k_p}$$

as an  $SL_2(K)$ -module. It is well known that if  $k_1 \geq 3$ , then  $K[W_{k_1}]$  contains nontrivial  $SL_2(K)$ -invariants. Similarly,  $K[W_2 \oplus W_2]^{SL_2(K)} \neq K$ . Hence the only cases when  $K[X_d]^{SL_2(K)} = K[(KX_d)^{SL_2(K)}]$  are  $k_1 = 2$ ,  $k_2 = \cdots = k_p = 1$  when  $K[X_d]^{SL_2(K)} = K[(KX_d)^{SL_2(K)}] \cong K[X_{d-1}]$  and  $k_1 = \cdots = k_p = 1$  with the trivial action of  $SL_2(K)$  on  $KX_d$  (and the latter case is impossible because  $G \cong SL_2(K)$  is a nontrivial subgroup of  $GL_d(K)$ ).

**3.3. Weitzenböck derivations.** A linear operator  $\delta$  of an algebra  $A$  is a *derivation* if

$$\delta(uv) = \delta(u)v + u\delta(v), \quad u, v \in A.$$

If  $\mathfrak{V}$  is a variety of algebras, then every mapping  $\delta : X_d \rightarrow F_d(\mathfrak{V})$  can be uniquely extended to a derivation of  $F_d(\mathfrak{V})$  which we shall denote by the same symbol  $\delta$ . If  $\delta$  is a nilpotent linear operator on  $KX_d$ , then the induced derivation is called a *Weitzenböck derivation*. Weitzenböck [36] proved that in the case of polynomial algebras the *algebra of constants*

$$K[X_d]^\delta = \{f(X_d) \in K[X_d] \mid \delta(f(X_d)) = 0\}$$

is finitely generated. Details on the algebra of constants  $K[X_d]^\delta$  can be found in the book by Nowicki [32]. For varieties  $\mathfrak{V}$  of unitary associative algebras (and  $\delta \neq 0$ ) the algebra  $F_d(\mathfrak{V})^\delta$  is finitely generated if and only if  $\mathfrak{V}$  does not contain the algebra  $T_2(K)$  of  $2 \times 2$  upper triangular matrices, see [15, 16]. Up to a change of the basis of  $KX_d$  the Weitzenböck derivation  $\delta$  is determined by the Jordan normal form  $J(\delta)$  of the linear operator  $\delta$  acting on  $KX_d$ . Since  $\delta$  acts nilpotently on  $KX_d$ , the matrix  $J(\delta)$  consists of Jordan blocks with zero diagonals.

**Proposition 3.5.** *If  $d > 2$  and the Jordan normal form  $J(\delta)$  of the Weitzenböck derivation consists of less than  $d - 1$  blocks, then the algebra  $F_d(\mathfrak{V})^\delta$  is not finitely generated for any variety  $\mathfrak{V}$  containing the variety  $\mathfrak{L}$ .*

*Proof.* Since  $\alpha\delta$ ,  $\alpha \in K$ , is nilpotent on  $KX_d$ , it is a *locally nilpotent derivation* of  $F_d(\mathfrak{V})$ , i.e., for every  $f(X_d) \in F_d(\mathfrak{V})$  there exists an  $n \geq 1$  such that  $(\alpha\delta)^n(f(X_d)) = 0$ . Hence

$$\exp(\alpha\delta) = 1 + \frac{\alpha\delta}{1!} + \frac{(\alpha\delta)^2}{2!} + \cdots$$

is a well defined linear automorphism of  $F_d(\mathfrak{V})$ . It is well known that the group

$$\{\exp(\alpha\delta) \mid \alpha \in K\}$$

is isomorphic to the unipotent group  $UT_2(K)$  and

$$F_d(\mathfrak{V})^\delta = F_d(\mathfrak{V})^{UT_2(K)}.$$

If the matrix  $J(\delta)$  consists of  $p$  blocks, then the dimension of the vector space  $(KX_d)^\delta$  of the linear constants is equal to the number of the blocks  $p$ . Reading carefully [32, Proposition 6.5.1, p. 65] we can see that

$$\text{transcend.deg}(K[X_d]^\delta) = d - 1$$

which is larger than  $p = \dim(KX_d)^\delta$ . Now the proof follows from Proposition 3.1 applied for  $UT_2(K) \subset GL_d(K)$ .  $\square$

**Remark 3.6.** If in Proposition 3.5 the Jordan normal form of  $\delta$  consists of  $d - 1$  blocks, then the algebra of constants  $K[X_d]^\delta$  is generated by linear constants. In this case we may assume that  $\delta(x_1) = x_2$  and  $\delta(x_i) = 0$  for  $i = 2, \dots, d$ . It is easy to see that  $F_d(\mathfrak{L})^\delta$  is generated by  $x_1x_2 - x_2x_1, x_2, \dots, x_d$ . We do not know how far can be lifted to  $F_d(\mathfrak{V})$  the finite generation property of the algebra of constants and do not have a description of the varieties  $\mathfrak{V}$  containing  $\mathfrak{L}$  such that the algebra  $F_d(\mathfrak{V})^\delta$  is finitely generated.

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